Leftist Grammars

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Outline

1. Motivations and definitions
2. Accessibility vs Grammars
3. Complexity of General Leftist Grammars
4. Restricted leftist grammars
5. Alphabet size restriction

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Accessibility Problems

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Accessibility problem

Can object p gain (illegal) access to object q by a series of legal moves?

Models for Accessibility Problems

1. access-matrix model: two-dimensional table of rights/priviliges; general rules for operations (undecidable);
2. grammatical protection systems: without creation of new objects;
3. take-grant model (limited expressive power).

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- a directed graph $G(V, E)$: V represents objects, E access rights;
- a type mapping $t : V \rightarrow T$: each vertex (object) has some label (type);
- binary relations $R_i, R_e \subseteq T \times T$: determine which operations are allowed.

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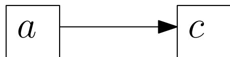
Restricted set of operations:

- Insert(a, x, c): combines original operations Insert and Give;
- Get(a, b, c): as before.

Operations (restricted)

$\mathrm{Insert}(a, x, c)$



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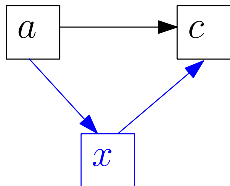

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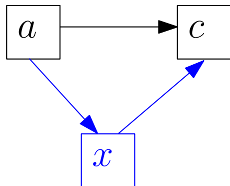

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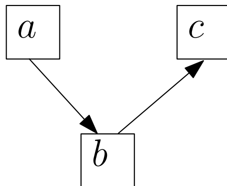

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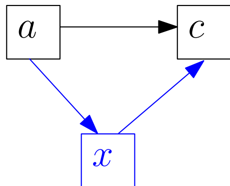

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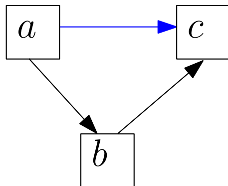

If $(t(b), t(c)) \in R_e$

Accessibility in the Cheiner/Saraswat Model

Accessibility Problem

Input:

- a directed graph $G(V, E)$;
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- $R_i, R_e \subseteq T \times T$.
- vertices $p, q \in V$.

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- a directed graph $G(V, E)$;
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Output:

YES iff there exists a sequence S of operations Insert, Get, Give such that the graph obtained from G after applying S contains the edge (p, q).

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- production rules P of the following two types,

$$
\begin{array}{rcll}
ab & \to & b & \text{(Delete Rule, corresponds to Get)} \\
c & \to & dc & \text{(Insert Rule, corresponds to Insert)}
\end{array}
$$

where $a, b, c, d \in \Sigma$.

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where $a, b, c, d \in \Sigma$.

A word $w \in \Sigma^*$ belongs to the language $L(\mathcal{G})$ iff $wx \Rightarrow^* x$.

Notations and Definitions

Shorter description

$$\begin{array}{lcllccl} a & \to & ba & \text{equivalent to} & a & \xrightarrow{\text{ins}} & b \\ ba & \to & a & \text{equivalent to} & a & \xrightarrow{\text{del}} & b. \end{array}$$

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Active symbol

Let

$$\begin{array}{l} u\ a\ v \Rightarrow u\ ba\ v \quad \text{or} \\ u\ ba\ v \Rightarrow u\ a\ v \end{array}$$

be a derivation step which applies a production $a \to ba$ (or $ba \to a$, resp.). Then, a is **active** in this step.

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Example

Alphabet

$\Sigma = \{a_0, a_1, b_0, b_1, x\}$

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Production rules

$b_1 \xrightarrow{\text{del}} a_1$

$b_0 \xrightarrow{\text{del}} a_0$

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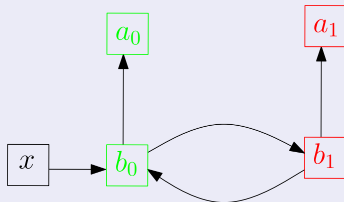
$b_0 \xrightarrow{\text{del}} b_1$

$x \xrightarrow{\text{del}} b_0$

Delete graph



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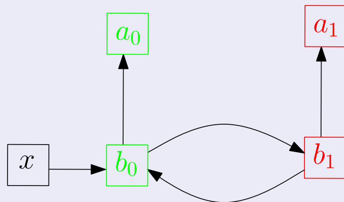


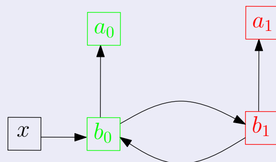
The language

$L(\mathcal{G}) \cap (a_0 a_1)^*(b_1 b_0)^* = \{(a_0 a_1)^n (b_1 b_0)^m \mid m \geq n\}$

Example cont.

a_0 a_1 a_0 $\underline{a_1}$ $\underline{b_1}$ b_0 b_1 b_0 x

Delete graph



Example cont.

$a_0 \quad a_1 \quad a_0 \quad \underline{a_1} \quad \underline{b_1} \quad b_0 \quad b_1 \quad b_0 \quad x$

$\Downarrow$

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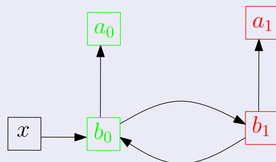

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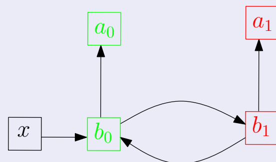

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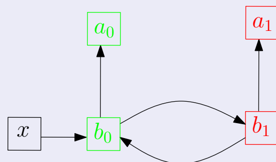

Example cont.

$$
\begin{array}{ccccccccc}
a_0 & a_1 & a_0 & \underline{a_1} & \underline{b_1} & b_0 & b_1 & b_0 & x \\
 & & & \Downarrow & & & & & \\
a_0 & a_1 & a_0 & & \underline{b_1} & \underline{b_0} & b_1 & b_0 & x \\
 & & & \Downarrow & & & & & \\
a_0 & a_1 & \underline{a_0} & & & \underline{b_0} & b_1 & b_0 & x \\
 & & & \Downarrow & & & & & \\
a_0 & a_1 & & & & \underline{b_0} & \underline{b_1} & b_0 & x \\
 & & & \Downarrow & & & & & \\
a_0 & \underline{a_1} & & & & & \underline{b_1} & b_0 & x
\end{array}
$$

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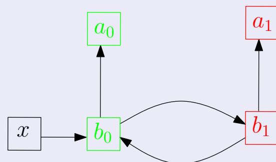

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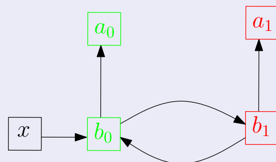

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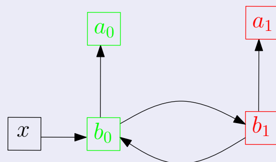

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			$\Downarrow$					
a_0	a_1	a_0		$\underline{b_1}$	$\underline{b_0}$	b_1	b_0	x
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a_0	a_1	$\underline{a_0}$			$\underline{b_0}$	b_1	b_0	x
			$\Downarrow$					
a_0	a_1				$\underline{b_0}$	$\underline{b_1}$	b_0	x
			$\Downarrow$					
a_0	$\underline{a_1}$					$\underline{b_1}$	b_0	x
			$\Downarrow$					
a_0						$\underline{b_1}$	$\underline{b_0}$	x
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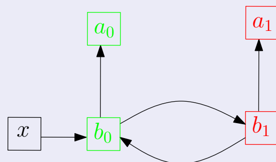

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$$x$$

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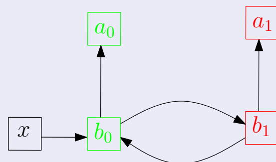

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Output: YES, iff $L(\mathcal{G}) \cap L(R) \neq \emptyset$.

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Membership problem (a leftist grammar $\mathcal{G}$ is fixed)

Input: a word w.
Output: YES, iff $wx \Rightarrow_{\mathcal{G}}^{*} x$.

Accessibility, Grammars, ...

Theorem

The accessibility problem in the Cheiner/Saraswat model can be reduced in polynomial time to the intersection problem for leftist grammars.

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Theorem (MPSV00)

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Corollary

The membership problem for leftist grammars is decidable.

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3. Closure under stuttering:

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How to deal with stuttering?

For a word $w \in (\Sigma_0 \cup \Sigma_1)^*$, where $\Sigma_0 \cap \Sigma_1 = \emptyset$, measure the number of color alternations, instead of the number of symbols.

Complexity of Leftist Grammars

MPSV (STOC00)

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The membership problem for leftist grammars is PSPACE-hard.

J. (ICALP08)

- variable membership problem: non-primitive recursive;
- ("static") membership problem: no primitive recursive upper bound.

Notations

Leftmost derivation

If

$$u \quad z \quad v \quad \Rightarrow \quad u \quad z' \quad v$$

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Notations

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Lemma

In order to verify wether $wx \Rightarrow^* x$, it is enough to consider greedy derivations.

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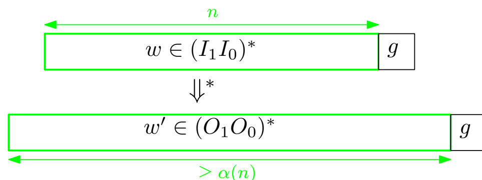

Weak computation: example

A grammar for the function $\alpha(n) = 2n$:

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Production rules

$$\begin{array}{ccl} b_i & \stackrel{\text{ins}}{\longrightarrow} & b_{(i+1) \bmod 4} \\ b_1 & \stackrel{\text{del}}{\longrightarrow} & a_0 \\ b_3 & \stackrel{\text{del}}{\longrightarrow} & a_1 \\ g & \stackrel{\text{ins}}{\longrightarrow} & b_0. \end{array}$$

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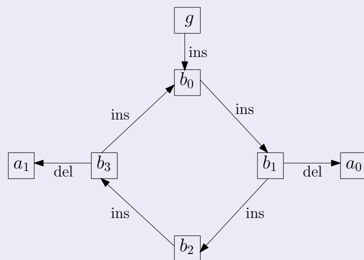
Insert/delete graph



Example: $\alpha(n) = 2n$

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$\Downarrow^*$

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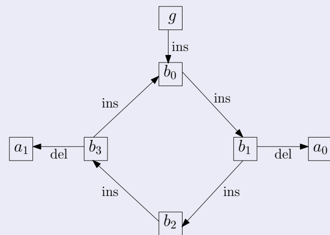

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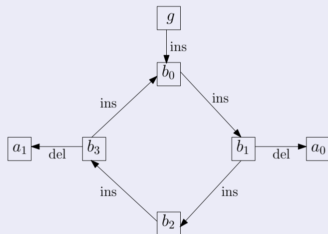

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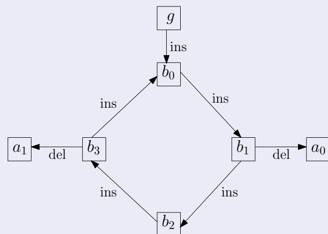

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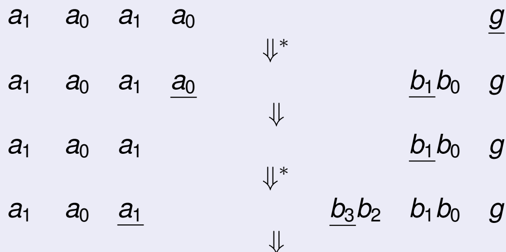

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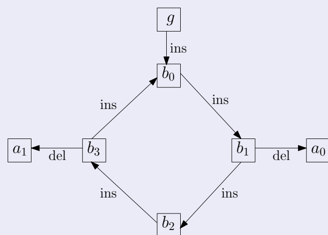

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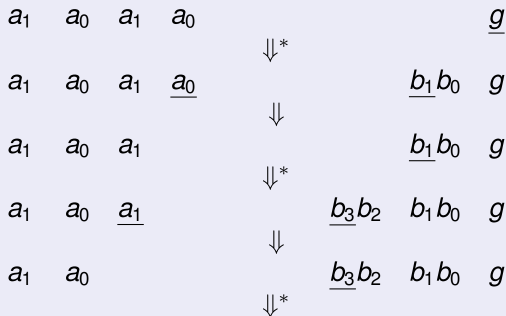

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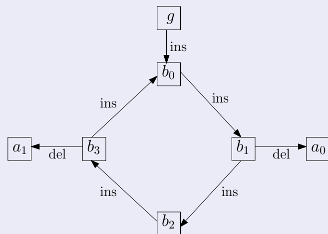

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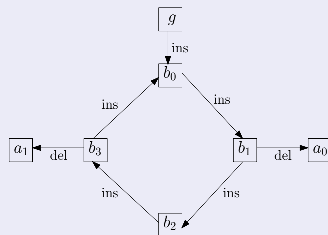

Exponentiation?

Exponentiation for the input subalphabets I_0, I_1

Build grammars $\mathcal{G}_0, \mathcal{G}_1$ such that:

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Claim

This grammar weakly computes $\alpha(n) = 2^n$ wrt I, $O_0 \cup O_1, g$ where $I = I_0 \cup I_1$.

Exponentiation: how does it work

$\imath_1$	$\imath_0$	$\imath_1$	$\imath_0$	g

$\imath_0 \in I_0 \quad \imath_1 \in I_1$

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$G_0 \Downarrow^*$

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$\imath_1$	$\imath_0$	$w_2 \in O_1^+$	g

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2^2

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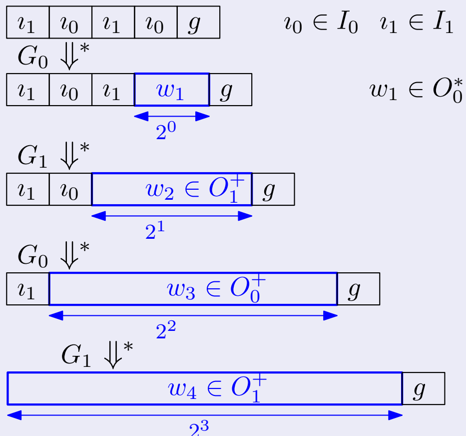

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Ackermann's function

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Inverse functions $(\phi_p^{-1})_{p \geq 2}$

$$\phi_p^{-1}(m) := \min\{n \mid \phi_p(n) \geq m\}.$$

Ackermann's Function

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For each $p \geq 2$, there exists a leftist grammar which weakly computes ϕ_p.

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is reduced to the problem

- Input: a leftist grammar $G(M)$ and an input word $w(M)$
- Output: yes iff $w(M) \in L(G(M))$.

Main reduction cont.

Reduction

Combine grammars:

Main reduction cont.

Reduction

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Check membership for $n = 3$.

Main reduction cont.

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Theorem

The variable membership problem for leftist grammars is non-primitive recursive.

Open problems – generalizations

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where $a, b, c, d \in \Sigma$.

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- Grammars with productions of types (1), (2), (3) or (1), (2), (4) ...?

Outline

1. Motivations and definitions
2. Accessibility vs Grammars
3. Complexity of General Leftist Grammars
4. Restricted leftist grammars
5. Alphabet size restriction

Restricted leftist grammars vs the Chomsky hierarchy

Restricted leftist grammars (J., Loryś ’05)

Delete graph	Insert graph	Included in	Not included in
acyclic	arbitrary	REG	FIN
arbitrary	empty	DCFL	REG
arbitrary	acyclic	CFL	DCFL
arbitrary	arbitrary	recursive*	SPACE(α)**

$\star$ – proved by MPSV

$\star\star$ – α is a primitive recursive function.

Open problems – restricted leftist grammars

Notation

LG(A, B):
languages defined by leftist grammars with delete graphs of type A and insert graphs of type B.

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1. Characterizations of classes
   LG(*acyclic*, *arbitrary*)
   LG(*arbitrary*, *empty*)
   LG(*arbitrary*, *acyclic*).
2. Comparison of these classes with REG_+, CFL_+, DCFL_+, where X_+ is the closure of the class X under stuttering.

Variable membership problem for restricted leftist grammars

Complexity of variable membership problem (J., '06, '08)

Delete graph	Insert graph	Lower bound	Upper bound
arbitrary	empty	?	EXPSPACE
acyclic	acyclic	PSPACE	EXPSPACE
acyclic	arbitrary	PSPACE	EXPSPACE
arbitrary	acyclic	PSPACE	EXPSPACE
arbitrary	arbitrary	non-prim.-rec.	recursive*

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$\star$ – proved by MPSV

Problems

1. Close gaps between lower and upper bounds.

Outline

1. Motivations and definitions
2. Accessibility vs Grammars
3. Complexity of General Leftist Grammars
4. Restricted leftist grammars
5. Alphabet size restriction

Leftist grammars with small alphabets

Assumption

For a grammar $\mathcal{G} = (\Sigma, P, x)$, productions of types

$$a \xrightarrow{\text{ins}} x$$
$$a \xrightarrow{\text{del}} x$$

are not allowed.

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S. Bandyopadhyay, M. Mahajan and K. N. Kumar (05)

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- Languages over two-letter alphabets defined by leftist grammars are regular.
- There exists a non-regular language over three-letter alphabet defined by a leftist grammar.

Small alphabets – problems

Conjecture

Languages over three-letter alphabets defined by leftist grammars are context-free.

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Languages over three-letter alphabets defined by leftist grammars are context-free.

Problems

1. Relationships between complexity of grammars and the size of the alphabet?

Small alphabets – problems

Conjecture

Languages over three-letter alphabets defined by leftist grammars are context-free.

Problems

1. Relationships between complexity of grammars and the size of the alphabet?
2. Exact characterizations of leftist languages over 2-letter, 3-letter, ... alphabets?

The End

Thank you for your attention!